5. Show that a necessary and sufficient condition that n solutions $\varphi_1, \varphi_2, \ldots, \varphi_n$ of differential equation :

$$a_0(t)\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n(t)y = 0,$$

$a_0(t) \neq 0$ to be linearly dependent on an interval I is that :

$$W(\varphi_1, \varphi_2, \ldots, \varphi_n) = 0 \text{ for all } t \in I.$$

Also show that every solution of differential equation is a suitable linear combination of n linearly independent solutions of given equation.

6. Determine the nature of the critical point (0, 0) of the system and determine whether or not the point is stable :

(a) $\frac{dx}{dt} = 2x - 7y; \frac{dy}{dt} = 3x - 8y$

(b) $\frac{dx}{dt} = 2x - 4y; \frac{dy}{dt} = 2x - 2y$.

S.C.No.—M/22/21703102

M.Sc. EXAMINATION, 2022

(Batch 2021) (First Semester)

MATHEMATICS

21MTH-102

Differential Equations and Calculus of Variations

Time : 3 Hours *Maximum Marks :* 80

Note : Attempt *Five* questions in all. All questions carry equal marks.

1. (a) State Sturm separation theorem.

(b) Define Lagrange's identity and Green's formula for second order.

(c) What do you mean by Pruffer transformation ?

(d) Define Fundamental Matrix.

(e) State and define classification of critical points.

(f) State Poincare-Bendixson theorem and define index of a critical point.

(g) Define Brachistochrone and Isoperimetric problem with suitable examples.

(h) State Euler's equation for one dependent function and its generalization to n dependent functions.

2. (a) Show that the function :

$$f(x, y) = x \sin y + y \cos x$$

satisfies Lipschitz condition in the rectangle D defined by $|x| \le a$, $|y| \le b$.

(b) Calculate the first three approximations to the solution of :

$$\frac{dy}{dt} = t^2 + y^2;\ y(0) = 0$$

By Picard's method.

3. (a) State and prove Sturms comparison theorem.

(b) Find the characteristic values and characteristic functions of the following Sturm-Liouville problem :

$$\frac{d^2y}{dx^2} + \lambda y = 0,\ y(0) = 0,\ y'(\pi) = 0.$$

4. (a) Show that the following equations is self-adjoint :

(i) $t^3 \frac{d^2x}{dt^2} + 3t^2 \frac{dx}{dt} + x = 0$

(ii) $\sin t \frac{d^2x}{dt^2} + \cos t \frac{dx}{dt} + 2x = 0.$

(b) Show that $\sin(t^3)$ and $\cos(t^3)$ is a fundamental set of the differential equation :

$$t\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 9t^5 x = 0$$

On every closed interval $[a, b]$, where $0 < a < b < \infty$.

7. (a) Solve the following system by Liapunov's method :

$$\frac{dx}{dt} = -x + y^2; \frac{dy}{dt} = -y + x^2$$

(b) Write short note on Floquet theory and limit cycles.

8. (a) State and prove first fundamental lemma of calculus of variations and also write the generalized form.

(b) Among all the curves joining two given points (x_0, y_0) and (x_1, y_1), find the one which generates the surface of minimum area when rotated about the x-axis.

9. (a) Find the extremals of the functional $J[y, z] = \int_0^{\pi/2} (y'^2 + z'^2 + 2yz)dx$ with boundary conditions $y(0) = z(0) = 0$, $y\left(\frac{\pi}{2}\right) = 1, z\left(\frac{\pi}{2}\right) = -1$.

(b) Find the geodesics of the circular cylinder $\vec{r} = (a\cos\phi, a\sin\phi, z)$.

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