(PG1-698) Roll No. ....

S.C.No.—21703104

M. Sc. EXAMINATION, 2023

(First Semester)

(Main/Re-appear/Improvement)

(2021/2022)

MATHEMATICS

21MTH-104

Real Analysis

Time : 3 Hours *Maximum Marks : 80*

Note : Attempt *Five* questions in all, selecting *one* question from each Unit. Q. No. **1** is compulsory. All questions carry equal marks.

Unit I

1. (a) Define Riemann Stieltjes integral.

(b) Define unit step function.

(c) Define Uniform convergence for series of functions.

(d) State Dirichlet's Test for Uniform convergence for series of functions.

(e) What is necessary condition for extreme value for a function ?

(f) Define Implicit Function.

(g) Prove that [0, 1] is uncountable.

(h) Define Lebesgue measurable set.

Unit II

2. (a) If $f_1, f_2 \in R(\alpha)$ on $[a, b]$, then prove that $f_1 + f_2 \in R(\alpha)$ and :

$$\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha.$$

(b) Let $f \in R(\alpha)$ on $[a, b]$, $m \le f \le M$, φ be a continuous function on $[m, M]$ and $h(x) = \varphi(f(x))$ on $[a, b]$. Then prove that $h \in R(\alpha)$ on $[a, b]$.

3. (a) Let f be Riemann integrable on $[a, b]$ and let there exist a differentiable function F such that $F'(x) = f(x)$ on $[a, b]$. Prove that $\int_a^b f(x)dx = F(b) - F(a)$.

(b) Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$ and α is continuous at every point at which f is discontinuous. Then prove that $f \in R(\alpha)$.

Unit-III

4. (a) Show that sequence given by $\{f_n\}$ where $f_n(x) = \tan^{-1} nx$, $x \geq 0$ is uniformly convergent in any interval $[a, b]$, $a > 0$ and is not uniformly convergent but pointwise convergent in $[0, b]$.

(b) If a series $\sum_{n=1}^{\infty} f_n$ of continuous functions is uniformly convergent to a function f on $[a, b]$, then prove that the sum function f is also continuous on $[a, b]$.

5. (a) Consider the series $\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}$. Test for uniform convergence and show it is integrable (term by term).

(b) State and prove Weierstrass Approximation Theorem.

Unit IV

6. (a) Let f maps an open set $E \subset R^n$ into R^m. Prove that f is continuous differentiable if and only if partial derivative $D_j f_i$ exists and are continuous on E for $1 \leq i \leq m$, $1 \leq j \leq n$.

(b) Prove that the function $f(x, y) = (|xy|)^{\frac{1}{2}}$ is not differentiable at point (0, 0) but f_x, f_y both exist at the origin and have the value zero. Hence deduce that these two partial derivatives are continuous except at origin.

7. (a) State and prove Inverse function theorem.

(b) Let $f(x, y, z) = 0$ be a functional relation where z is dependent variable such that $z = z(x, y)$. Find the partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ in terms of p, q, r, s, t.

Unit V

8. (a) If $A \subseteq B$, then prove that :

$$m^*(A) \leq m^*(B).$$

(b) Prove that Outermeasure is countably sub-additive.

9. (a) Prove that the family M of all measurable sets is a σ-algebra.

(b) Construct a non-measurable set in [0, 1).